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COMMENT

Exact multi-soliton solution of the Benjamin–Ono equation

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Abstract. An exact multi-soliton solution of the Benjamin–Ono equation is presented. The asymptotic form of the solution for large time is also given.

The Benjamin–Ono equation (Benjamin 1966, 1967, Ono 1975) which describes internal waves is written as

$$\frac{\partial u(x,t)}{\partial t}+4u(x,t)\frac{\partial u(x,t)}{\partial x}+H\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)=0 \tag{1}$$

where H is the Hilbert transform defined by

$$H[u(x,t)]=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(y,t)}{y-x}\,\mathrm{d}y. \tag{2}$$

The stationary solution of equation (1) is given by

$$u(x,t)=\frac{V}{V^2(x-Vt-\xi)^2+1} \tag{3}$$

where V and ξ are arbitrary constants which may be called the velocity and the phase respectively.

Recently Joseph (1977) proposed an analytical method for solving equation (1) and obtained a two-soliton solution explicitly. Meiss and Pereira (1978) found new conserved quantities of equation (1) and surmised the existence of exact two- and three-soliton solutions.

In the present comment we solve equation (1) by transforming it into a bilinear form according to Hirota's method (Hirota 1971) and give an explicit expression for the N-soliton solution of equation (1).

Now we seek a solution of equation (1) which is real and finite over all x, t and express it in the following form:

$$u(x,t)=\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\ln\frac{f^*(x,t)}{f(x,t)} \tag{4}$$

$$f(x,t)\propto(x-x_1(t))(x-x_2(t))\ldots(x-x_N(t)) \tag{5}$$

$$\operatorname{Im} x_n>0 \qquad n=1,2,\ldots,N \tag{6}$$

where x_n $(n=1,2,\ldots,N)$ are complex functions of t and * denotes a complex conjugate.

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Using (4)–(6) and the formula

$$P\frac{1}{y-x}=\tfrac{1}{2}\lim_{\epsilon\to+0}\left(\frac{1}{y-x+i\epsilon}+\frac{1}{y-x-i\epsilon}\right) \tag{7}$$

we obtain

$$\begin{aligned} H[u(x,t)] &= iu(x,t)-\sum_{n=1}^{N}\frac{1}{x-x_n} \\ &= iu(x,t)-\frac{\partial f(x,t)/\partial x}{f(x,t)} \end{aligned} \tag{8}$$

where we have performed the contour integral closed by a large semicircle in the upper half-plane and used

$$\lim_{x\to x_n}[(x-x_n)u(x,t)]=\frac{1}{2i} \qquad n=1,2,\ldots,N \tag{9}$$

which are derived from (4) and (5).

Substituting (4) and (8) into equation (1), we get a bilinear equation for f as follows:

$$\mathrm{Im}(f_t^* f)=f_x^* f_x-\mathrm{Re}(f_{xx}^* f) \tag{10}$$

where subscripts x, t denote partial differentiations. The solution of equation (10) is expressed as

$$f_N=\det M \tag{11}$$

where M is an $N\times N$ matrix whose elements are given by

$$M_{nm}=\begin{cases} i\theta_n+1 & \text{for } n=m \\ \dfrac{2(V_nV_m)^{1/2}}{V_n-V_m} & \text{for } n\neq m, \end{cases} \tag{12}$$

with

$$\theta_n=V_n(x-V_nt-\xi_n) \tag{13}$$

where V_n and ξ_n $(n=1,2,\ldots,N)$ are arbitrary constants and it is assumed that $V_n\neq V_m$ for $n\neq m$. It is confirmed by direct substitution that (11)–(13) give an exact solution of equation (10). The equivalence of the present results with known solutions for one- and two-soliton solutions can be checked easily.

For $N=1$, we have

$$f_1=i\theta_1+1. \tag{14}$$

Substitution of (14) into (4) gives a one-soliton solution (3) immediately.

For $N=2$, we have

$$f_2=-\theta_1\theta_2+i(\theta_1+\theta_2)+V_{12} \tag{15}$$

where

$$V_{nm}=[(V_n+V_m)/(V_n-V_m)]^2. \tag{16}$$

This solution corresponds to a two-soliton solution obtained by Joseph (1977). The equivalence of (4) substituted from (15) to Joseph's solution (2.63) can easily be seen by transforming his equation (1.4) into a moving frame with a velocity c_0 and putting

$$C=4 \qquad c_0\gamma=1 \qquad p=(V_1^2+V_2^2-4V_1V_2)/V_1V_2 \qquad q=\frac{1}{(V_1V_2)^{1/2}}\frac{V_2+V_1}{V_2-V_1}.$$

For $N=3$, we have

$$f_3=-\mathrm{i}\theta_1\theta_2\theta_3-(\theta_1\theta_2+\theta_2\theta_3+\theta_3\theta_1)$$
$$+\mathrm{i}(V_{23}\theta_1+V_{31}\theta_2+V_{12}\theta_3)+V_{12}+V_{23}+V_{31}-2, \tag{17}$$

which corresponds to a three-soliton solution.

Now we investigate the asymptotic behaviour of our solution for large t. From (4), (11), (12) and (13), we get for $t\to\pm\infty$

$$u(x,t)\sim\sum_{n=1}^{N}\frac{V_n}{V_n^2(x-V_nt-\xi_n)^2+1}, \tag{18}$$

that is, $u(x, t)$ is represented by a superposition of one-soliton solutions (3). It is seen from (18) that *no phase-shift appears as the result of collisions of solitons* unlike those which take place between K–dV solitons (Gardner *et al* 1974). This is an interesting feature of the system of solutions expressed by (11)–(13).

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